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On the one-dimensional Coulomb Hamiltonian

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Abstract. We give a rigorous definition of the one-dimensional Coulomb Hamiltonian, discuss its spectral properties, and investigate various approximations for it. For one of these approximations, which is frequently used in the literature, we prove in particular that it converges to the semi-bounded Coulomb Hamiltonian in the strong graph limit, although its ground-state energy tends to minus infinity in this limit.

1. Introduction

Recently Mehta and Patil (1978) and van Haeringen (1978) investigated modified Coulomb interactions of the form

$$V_\alpha(x) = \frac{c}{|x|+\alpha} \qquad \alpha > 0 \tag{1.1}$$

which correspond to a smeared charge distribution (rather than a point charge) and thus may be useful for the description of mesic atoms. For this potential Mehta and Patil (1978) proved dispersion relations and an approximation (as $\alpha \to 0_+$) for the s-wave bound states, and van Haeringen (1978) discussed the bound states of the corresponding one-dimensional Schrödinger operator. The aim of this paper is to give a rigorous description of the one-dimensional Coulomb Hamiltonian H (see equations (2.6) and (2.7) below) and its spectral properties. We also discuss two approximations, involving the potential (1.1), which we prove to converge to H in the norm, or respectively in the strong resolvent sense as $\alpha \to 0_+$. (The s-waves of the three-dimensional problem are obviously contained in our treatment.) In particular, we prove that the strong graph (resolvent) limit of the approximation T_α (see equations (3.1) and (3.2)) used by van Haeringen (1978) and Loudon (1959) converges to H. Since $E_0(\alpha)$, the ground-state energy of this approximation T_α, behaves like

$$E_0(\alpha) \approx -c^2 \ln^2(-2c\alpha) \qquad |c\alpha| \ll 1 \tag{1.2}$$

and thus diverges as $\alpha \to 0_+$, it was sometimes conjectured that the one-dimensional Coulomb Hamiltonian H has no ground state (Loudon 1959, van Haeringen 1978). In contrast to this conjecture we prove that H is bounded from below by

$$H \geqslant -c^2/4 \tag{1.3}$$

with $-c^2/4$ being the ground-state energy of H. The main feature of functions

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contained in the domain of H is that they obey a Dirichlet boundary condition at the origin due to the singular character of $1/|x|$. (For similar kinds of operators see Harrell (1977) and references therein.)

2. The Coulomb Hamiltonian

Let

$$V(x)=c/|x| \qquad c\in R, c\neq 0 \tag{2.1}$$

and define

$$\dot{H}=-\frac{d^2}{dx^2}+V(x) \qquad \text{on } D(\dot{H})=C_0^\infty(R-\{0\}). \tag{2.2}$$

Then $\bar{\dot{H}}$, the closure of $\dot{H}$, is given by

$$\bar{\dot{H}}=-\frac{d^2}{dx^2}+V(x) \qquad D(\bar{\dot{H}})=\{f|f,f'\in A_{\text{loc}}(R); f(0)=f'(0)=0; f,f',f''\in L^2(R)\} \tag{2.3}$$

i.e.

$$\bar{\dot{H}}=H_+\oplus H_-+V \tag{2.4}$$

where

$$H_\pm=-\frac{d^2}{dx^2} \qquad D(H_\pm)=\{f|f,f'\in A_{\text{loc}}(R_\pm); f(0_\pm)=f'(0_\pm)=0; f,f',f''\in L^2(R_\pm)\}. \tag{2.5}$$

$A_{\text{loc}}(a,b)$ denotes the set of locally absolutely continuous functions on (a,b) and $R_\pm$ denotes the interval $R_\pm=(0,\pm\infty)$.

We shall see, by virtue of (2.10), that $\bar{\dot{H}}$ is well defined, semi-bounded and closed but not self-adjoint. As a natural self-adjoint extension we take the Friedrichs extension H of $\dot{H}$:

$$H=H_D+V \qquad D(H)=D(H_D) \tag{2.6}$$

where H_D is the Friedrichs extension of $H_+\oplus H_-$ (with a Dirichlet boundary condition at the origin):

$$H_D=-\frac{d^2}{dx^2} \qquad D(H_D)=\{f|f,f'\in A_{\text{loc}}(R-\{0\}); f(0)=0; f,f',f''\in L^2(R)\} \tag{2.7}$$

(i.e. $H_D=H_{+F}\oplus H_{-F}$). Here $f'\in A_{\text{loc}}(R-\{0\})$ means that $f'(x)\in A_{\text{loc}}(0,\infty)$ for $x>0$ and $f'(x)\in A_{\text{loc}}(-\infty,0)$ for $x<0$, but possibly $f'(0_+)\neq f'(0_-)$; $f(0)=0$ is to be interpreted as $f(0_+)=f(0_-)=0$. To show that H is well defined we note that from Hardy's inequality

$$\frac{1}{2}\left\|\frac{1}{x}f\right\|\leq\|f'\|\leq\|f'\|_+ + \|f'\|_- \qquad f(0)=0; f,f'\in L^2(R) \tag{2.8}$$

(the subscripts refer to the norms in $L^2(R_\pm)$, respectively), together with the fact that d/dx is infinitesimally bounded with respect to d^2/dx^2 in $L^2(R_\pm)$, i.e.

$$\|f'\|_\pm \leq \epsilon\|f''\|_\pm + \frac{2}{\epsilon}\|f\|_\pm \qquad \epsilon > 0; f, f', f'' \in L^2(R_\pm) \tag{2.9}$$

we infer

$$\|Vf\| \leq \epsilon\|H_D f\| + \frac{2}{\epsilon}\|f\| \qquad \epsilon > 0; f \in D(H_D). \tag{2.10}$$

Thus V is infinitesimally bounded with respect to H_D.

Note that for the s-wave Hamiltonian in the three-dimensional problem one automatically takes the Friedrichs extension (characterised by the boundary condition $f(0_+) = 0$) of

$$H_+ + V(r) \qquad \text{in} \qquad L^2(R_+)$$

since $-\Delta + V(|\boldsymbol{x}|)$ is not essentially self-adjoint on $C_0^\infty(R^3 - \{0\})$.

Next we turn to the sesquilinear forms h, h_D and h_V corresponding to H, H_D and V:

$$h(f, g) = h_D(f, g) + h_V(f, g) \qquad D(h) = D(h_D) \tag{2.11}$$

$$h_D(f, g) = (f', g') \qquad D(h_D) = \{f | f \in A_{\text{loc}}(R); f(0) = 0; f, f' \in L^2(R)\} \tag{2.12}$$

$$h_V(f, g) = \int_R dx\, V(x)\bar{f}(x)g(x) \qquad D(h_V) = \left\{ f \in L^2(R) \middle| \int_R dx\, V(x)|f(x)|^2 < \infty \right\}. \tag{2.13}$$

Clearly, h_V is infinitesimally bounded with respect to h_D and

$$h = \overline{h|_{C_0^\infty(R-\{0\})}}.$$

Define

$$V_\alpha(x) = \frac{c}{|x| + \alpha} \qquad \alpha > 0, c \in R, c \neq 0 \tag{2.14}$$

and introduce

$$H_\alpha = H_D + V_\alpha \qquad D(H_\alpha) = D(H_D) \tag{2.15}$$

$$h_\alpha(f, g) = h_D(f, g) + h_{V_\alpha}(f, g) \qquad D(h_\alpha) = D(h_D) \tag{2.16}$$

$$h_{V_\alpha}(f, g) = (f, V_\alpha g) \qquad D(h_{V_\alpha}) = L^2(R). \tag{2.17}$$

We shall investigate the limit of H_α as $\alpha \to 0_+$ and its spectral properties. Let us denote by $R(A, z)$ the resolvent $(A - z)^{-1}$, by $\rho(A)$ the resolvent set, by $\sigma(A)$ the spectrum, by $\sigma_{\text{ess}}(A)$ the essential spectrum, by $\sigma_{\text{ac}}(A)$ the absolutely continuous spectrum, by $\sigma_{\text{p}}(A)$ the point spectrum, and by $\sigma_{\text{d}}(A)$ the discrete spectrum of a self-adjoint operator A. Then we have:

Lemma 2.1. (a) If $c > 0$ then

$$\sigma(H_\alpha) = \sigma_{\text{ac}}(H_\alpha) = [0, \infty) \tag{2.18}$$

and H_α has no eigenvalues,

$$\sigma_{\text{p}}(H_\alpha) = \varnothing. \tag{2.19}$$

(b) If $c<0$ then

$$H_\alpha \geqslant \max(-c^2/4, c/\alpha) \tag{2.20}$$

and

$$\sigma_{\mathrm{ess}}(H_\alpha)=\sigma_{\mathrm{ac}}(H_\alpha)=[0,\infty). \tag{2.21}$$

The point spectrum $\sigma_{\mathrm{p}}(H_\alpha)$ is purely discrete; it consists of infinitely many isolated eigenvalues, each of which has multiplicity two. For $|c\alpha| \ll 1$ it is given by (Loudon 1959, Mehta and Paril 1978, van Haeringen 1978)

$$\sigma_{\mathrm{p}}(H_\alpha)=\sigma_{\mathrm{d}}(H_\alpha)=\left\{E_n \middle| E_n=-\frac{c^2}{4n^2}\left(1+\frac{2c\alpha}{n}-\frac{2c^2\alpha^2\ln\alpha}{n}\right)+\mathrm{O}(\alpha^3),\ n=1,2,\ldots\right\}. \tag{2.22}$$

Proof. Suppose ψ is an eigenvector of H_α to the eigenvalue $E(\alpha)$:

$$H_\alpha\psi=E(\alpha)\psi.$$

Then, using scale transformations one proves the virial theorem

$$2(\psi, H_D\psi)=-\frac{1}{c}(\psi, |x|V_\alpha^2\psi)=2(\psi,[E(\alpha)-V_\alpha]\psi). \tag{2.23}$$

Let $c>0$. Then (2.23) implies

$$2(\psi, H_D\psi)=-\frac{1}{c}(\psi, |x|V_\alpha^2\psi)\leqslant 0.$$

Hence $\psi=0$ and (2.19) holds. To prove (2.18) we only note that H_α is the orthogonal sum of two self-adjoint operators, acting on $L^2(R_+)$ and $L^2(R_-)$, respectively, whose spectra are purely absolutely continuous and cover $[0,\infty)$ (Weidmann 1967). Let $c<0$. Then (2.23) implies

$$2E(\alpha)\|\psi\|^2=2(\psi, V_\alpha\psi)-\frac{1}{c}(\psi, |x|V_\alpha^2\psi)\leqslant(\psi, V_\alpha\psi)<0$$

and hence $E(\alpha)<0$. If $E<0$, orthogonal solutions of

$$-\psi''(x)+\frac{c}{|x|+\alpha}\psi(x)=E\psi(x)$$

are given by

$$\psi^{(1)}(x)=\begin{cases} N\exp[-\sqrt{-E}(x+\alpha)](x+\alpha)U\left(1+\frac{c}{2\sqrt{-E}};2;2\sqrt{-E}(x+\alpha)\right) & x\geqslant 0 \\ 0 & x\leqslant 0 \end{cases}$$

$$\psi^{(2)}(x)=\begin{cases} 0 & x\geqslant 0 \\ N\exp[-\sqrt{-E}(-x+\alpha)](-x+\alpha)U\left(1+\frac{c}{2\sqrt{-E}};2;2\sqrt{-E}(-x+\alpha)\right) & x\leqslant 0 \end{cases} \tag{2.24}$$

where $U(a;b;z)$ is the irregular confluent hypergeometric function and N is some normalisation constant. The eigenvalues E_n are determined by the equation

$$U\left(1+\frac{c}{2\sqrt{-E}}; 2; 2\sqrt{-E}\alpha\right)=0. \tag{2.25}$$

For $|c\alpha| \ll 1$ one can expand U to obtain the approximation (2.22) (cf Loudon 1959, Mehta and Patil 1978, van Haeringen 1978). From (van Haeringen 1978)

$$\Gamma\left(1+\frac{c}{2\sqrt{-E}}\right)2\alpha\sqrt{-E}U\left(1+\frac{c}{2\sqrt{-E}}; 2; 2\sqrt{-E}\alpha\right)$$
$$=\int_0^\infty \mathrm{d}t \exp(-2\alpha\sqrt{-E}t)\left[1+\frac{c}{2\sqrt{-E}}+2\alpha\sqrt{-E}(1+t)\right]$$
$$\times t^{c/2\sqrt{-E}}(1+t)^{-2-c/2\sqrt{-E}}>0 \qquad \text{for} \qquad c/2\sqrt{-E}>-1$$

one recognises that there are no zeros of (2.25) for $E<-c^2/4$. Since obviously $H_\alpha \geq c/\alpha$, (2.20) is proved. The result (2.21) again follows from the fact that H_α is an orthogonal sum of two self-adjoint operators whose essential spectra are absolutely continuous and cover $[0, \infty)$.

Remark 2.1. (Mehta and Patil 1978). Let $\lambda>0$ and $U(\lambda)$ be the unitary transformation which implements scaling:

$$(U(\lambda)f)(x)=\lambda^{-1/2}f(\lambda^{-1}x) \qquad f\in L^2(R).$$

Then $U(\lambda)$ leaves $D(H_D)$ invariant and, if we write $H_\alpha(c)$ to exhibit explicitly the c dependence of H_α, we obtain

$$U(\lambda)H_\alpha(c)U(\lambda)^{-1}=\lambda^2 H_{\lambda\alpha}(c/\lambda)$$

and consequently

$$E_n(c, \alpha)=\lambda^2 E_n(c/\lambda, \lambda\alpha). \tag{2.26}$$

Next we consider the convergence of H_α to H.

Lemma 2.2. Let $z\in\rho(H)$. Then $z\in\rho(H_\alpha)$ for α sufficiently small and $R(H_\alpha, z)$ converges to $R(H, z)$ in norm as $\alpha\to 0_+$:

$$\lim_{\alpha\to 0_+}\|R(H_\alpha, z)-R(H, z)\|=0. \tag{2.27}$$

Proof.

$$|h_\alpha(f,f)-h(f,f)|$$
$$=\left|\left(f, \frac{c\alpha}{|x|(|x|+\alpha)}f\right)\right|\leq\alpha|c|\int_R \mathrm{d}x\frac{|f(x)|^2}{|x|^2}\leq 4\alpha|c|h_D(f,f), \qquad f\in D(h_D)$$

with the help of (2.8). Now

$$h_D(f,f)=h(f,f)-h_V(f,f)\leq h(f,f)+|c|\left(f, \frac{1}{|x|}f\right)$$
$$\leq h(f,f)+|c|\epsilon h_D(f,f)+|c|\eta(\epsilon)\|f\|^2 \qquad \epsilon, \eta(\epsilon)>0.$$

Thus

$$|h_\alpha(f,f)-h(f,f)|$$

$$\leqslant 4\alpha|c|h_D(f,f)\leqslant \frac{4\alpha|c|}{1-|c|\epsilon}h(f,f)+\frac{4\alpha c^2\eta(\epsilon)}{1-|c|\epsilon}\|f\|^2 \qquad \epsilon>0, f\in D(h_D)$$

and we only need to apply Kato (1966, Theorem VI 3.6). (For detailed information about norm (strong) resolvent convergence compare Kato (1966), Reed and Simon (1972), Schechter (1976) and Simon (1978).)

Now we turn to the operator H and state

Theorem 2.1. (a) Let $c>0$. Then

$$\sigma(H)=\sigma_{\mathrm{ac}}(H)=[0,\infty) \tag{2.28}$$

$$\sigma_{\mathrm{p}}(H)=\varnothing. \tag{2.29}$$

(b) If $c<0$, then

$$\sigma_{\mathrm{ess}}(H)=\sigma_{\mathrm{ac}}(H)=[0,\infty) \tag{2.30}$$

$$\sigma_{\mathrm{p}}(H)=\sigma_{\mathrm{d}}(H)=\left\{E_n|E_n=-\frac{c^2}{4n^2}, n=1,2,\ldots\right\} \tag{2.31}$$

and each eigenvalue has multiplicity two. The corresponding eigenspace is spanned by the functions

$$\begin{aligned}\psi_n^{(1)}(x)&=N'\exp\left(\frac{c}{2n}|x|\right)2xU\left(1-n;2;-\frac{c}{n}|x|\right) \qquad n=1,2,\ldots\\ \psi_n^{(2)}(x)&=N'\exp\left(\frac{c}{2n}|x|\right)|x|U\left(1-n;2;-\frac{c}{n}|x|\right) \qquad n=1,2,\ldots.\end{aligned} \tag{2.32}$$

Proof. If ψ is an eigenvector of H to the eigenvalue E the virial theorem reads

$$2(\psi, H_D\psi)=-(\psi, V\psi)=2(\psi,[E-V]\psi). \tag{2.33}$$

Let $c>0$. Then (2.33) implies $(\psi, H_D\psi)\leqslant 0$ from which we conclude $\psi=0$ and thus (2.29) is established. The result (2.28) follows as in the proof of lemma 1 ($\sigma(H)=[0,\infty)$ already follows from (2.18) and (2.27)). Now we turn to $c<0$. If ψ is an eigenfunction of H, (2.33) yields

$$2E\|\psi\|^2=(\psi, V\psi)<0,$$

hence $E<0$. By inspection it is easily verified that $E_n=-c^2/4n^2$, $n=1,2,3,\ldots$, are eigenvalues of multiplicity two and the functions of (2.32) are the corresponding eigenfunctions. (From (2.22) and strong resolvent convergence of H_α to H it follows that these E_n are the only negative numbers contained in $\sigma(H)$.) The result (2.30) is proved by the same arguments used in the proof of lemma 1.

3. The approximation T_α

From lemmas 2.1 and 2.2 one concludes that H_α approximates H in a very natural manner. Despite this fact the approximations normally used (Loudon 1959, van

Haeringen 1978) are quite different from H_α. Usually one introduces the sequence $\{T_\alpha\}$

$$T_\alpha = T + V_\alpha \qquad D(T_\alpha) = D(T) \tag{3.1}$$

where

$$T = -\frac{d^2}{dx^2} \qquad D(T) = \{f | f, f' \in A_{loc}(R); f, f', f'' \in L^2(R)\}. \tag{3.2}$$

As α tends to 0_+ the ground state energy $E_0(\alpha)$ of T_α tends to $-\infty$ (see lemma 3.1 below) and this was the reason why it was frequently conjectured (Loudon 1959, van Haeringen 1978) that the one-dimensional hydrogen Hamiltonian is not bounded from below. Because of theorem 2.1 this is obviously wrong and we shall prove that the strong graph limit of T_α equals H or, equivalently, the resolvent of T_α converges strongly to the resolvent of H as $\alpha \to 0_+$. For that purpose we first give a description of $\sigma(T_\alpha)$:

Lemma 3.1. (a) For $c > 0$ we have

$$\sigma(T_\alpha) = \sigma_{ac}(T_\alpha) = [0, \infty) \tag{3.3}$$

$$\sigma_p(T_\alpha) = \varnothing. \tag{3.4}$$

(b) If $c < 0$ then

$$\sigma_{ess}(T_\alpha) = \sigma_{ac}(T_\alpha) = [0, \infty) \tag{3.5}$$

and the point spectrum of T_α consists of infinitely many isolated non-degenerate eigenvalues. For $|c\alpha| \ll 1$ they are approximately given by (Loudon 1959, van Haeringen 1978)

$$\begin{aligned} E_0^{(e)} &\approx -c^2 \ln^2(-2c\alpha) \\ E_n^{(e)} &\approx -\frac{c^2}{4n^2}\left(1 + \frac{2}{n \ln(-c\alpha)}\right) \qquad n = 1, 2, \ldots \\ E_n^{(o)} &\approx -\frac{c^2}{4n^2}\left(1 + \frac{2c\alpha}{n}\right) \qquad n = 1, 2, \ldots \end{aligned} \tag{3.6}$$

where (e) and (o) refer to the corresponding even and odd eigenfunctions, respectively.

Proof. Suppose ψ to be an eigenvector of T_α with eigenvalue E. Then we have the virial theorem

$$2(\psi, T\psi) = -\frac{1}{c}(\psi, |x| V_\alpha^2 \psi) = 2(\psi, [E - V_\alpha]\psi). \tag{3.7}$$

Let $c > 0$. Then (3.4) follows from (3.7) as in the proof of lemma 1.

Since $V_\alpha \in L^2(R)$ we have for all c

$$\sigma_{ess}(T_\alpha) = \sigma_{ess}(T) = [0, \infty).$$

The absolute continuity of the essential spectra follows from Weidmann (1967). That there are infinitely many eigenvalues for $c < 0$ already follows from the existence of $A > 0$, $R > 0$, such that (Weidmann 1976)

$$V_\alpha(x) \leqslant -A/|x| \qquad \text{for} \qquad |x| \geqslant R.$$

Non-degeneracy of the eigenvalues follows in the usual manner. Let ψ_1 and ψ_2 be two eigenfunctions to a given eigenvalue E. Then $\psi_1, \psi_2 \in D(T)$ implies $\psi_1'\psi_2 - \psi_1\psi_2' \in L^1(R)$ and so $\psi_1'\psi_2 - \psi_1\psi_2' = 0$ using Schrödinger's equation. Thus $\psi_1 = \text{constant} \times \psi_2$.

Finally the expressions (3.6) are obtained from the eigenvalue conditions for even and odd solutions, respectively,

$$U\left(\frac{c}{2\sqrt{-E}}; 0; 2\sqrt{-E}\alpha\right) = 2U\left(\frac{c}{2\sqrt{-E}}; 1; 2\sqrt{-E}\alpha\right)$$

and

$$U\left(\frac{c}{2\sqrt{-E}}; 0; 2\sqrt{-E}\alpha\right) = 0. \tag{3.8}$$

Since the scale transformation $U(\lambda)$ leaves $D(T)$ invariant too, (2.26) remains valid for all eigenvalues of T_α. To prove convergence of T_α to H in the strong graph (resolvent) sense we first state a criterion due to Wüst (1973):

Lemma 3.2. Suppose

(a) $\{A_\alpha\}$, $\alpha \in (0, \alpha_0]$, $\alpha_0 > 0$ is a family of self-adjoint operators in a Hilbert space H and $D(A_\alpha) = D$ for all $\alpha \in (0, \alpha_0]$;

(b) $A_\alpha - A_\beta \geq 0$ for all $\alpha, \beta \in (0, \alpha_0]$ with $\alpha \geq \beta$ (or $\beta \geq \alpha$);

(c) $\overline{(A_\alpha - A_\beta)}$ is bounded for all $\alpha, \beta \in (0, \alpha_0]$ and

$$\lim_{\alpha \to \beta} \|\overline{(A_\alpha - A_\beta)}\| = 0 \qquad \text{for all } \alpha, \beta \in (0, \alpha_0];$$

(d) there exist constants $\lambda \in R$, $\mu > 0$ such that

$$\|(A_\alpha - \lambda)f\| \geq \mu \|f\| \qquad \alpha \in (0, \alpha_0], f \in D;$$

(e) $\{A_\alpha\}$ is strongly graph convergent to an operator A_0 as $\alpha \to 0_+$.

Then the resolvents of A_α converge strongly to the resolvent of A_0:

$$\text{s-lim}_{\alpha \to 0_+} R(A_\alpha, z) = R(A_0, z) \qquad \text{Im } z \neq 0$$

and the graph limit A_0 is self-adjoint. Thus $D(A_0)$ is given by

$$D(A_0) = \{f \in H \mid \text{there are } f_\alpha \in D\text{: } \lim_{\alpha \to 0_+} \|f_\alpha - f\| = 0 \text{ and } \text{s-lim}_{\alpha \to 0_+} A_\alpha f_\alpha \text{ exists}\}.$$

(For graph limits see Glimm and Jaffe (1969), Reed and Simon (1972, 1975) and Wüst 1973.) This result applied to the family $\{T_\alpha\}$ yields

Theorem 3.1. The operator H is the strong graph limit of T_α as $\alpha \to 0_+$

$$\text{s-graph-lim}_{\alpha \to 0_+} T_\alpha = H \tag{3.9}$$

or, equivalently, for Im $z \neq 0$, $R(T_\alpha, z)$ converges strongly to $R(H, z)$:

$$\text{s-lim}_{\alpha \to 0_+} R(T_\alpha, z) = R(H, z) \qquad \text{Im } z \neq 0. \tag{3.10}$$

Proof. We have to check (a)–(e) of lemma 3.2. By definition (a) and (b) are fulfilled. V_α is bounded for $\alpha > 0$, hence (c) is true. If $c > 0$ then (d) is valid because of $T_\alpha \geqslant 0$. For $c < 0$ and $|c\alpha| \ll 1$ there is a gap between $E_0^{(e)}$ and $-c^2/4$ by lemma 3.1 and hence (d) is fulfilled in this case too. By the dominated convergence theorem we have $\text{s-lim}_{\alpha \to 0_+} T_\alpha f = Hf$ for $f \in D(H_D) \cap D(T)$, hence (e) is true. By lemma 3.2 the strong graph limit of T_α is self-adjoint. It clearly coincides with H. Since T_α, H are self-adjoint, strong graph convergence is equivalent to strong resolvent convergence, which completes the proof.

The results of lemma 3.1 and theorem 3.1 are certainly not restricted to the special choice $V_\alpha(x) = c/(|x| + \alpha)$; they still hold if some similar bounded approximation $\tilde{V}_\alpha(x)$ is used (e.g. the cut-off approximation to be found in Loudon (1959)).

From theorem 3.1 one infers that in going from T_α to H one encounters a Dirichlet boundary condition at zero and in addition the ground-state energy $E_0^{(e)}(\alpha)$ disappears in the limit $\alpha \to 0_+$, i.e. it is not contained in the spectrum of H. Nevertheless the approximation T_α converges to the Coulomb Hamiltonian H in a reasonable sense for $\alpha \to 0_+$ and all of its other eigenvalues converge to the corresponding ones of H.

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